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SOME REMARK OF A FUNCTION REPRESENTED BY INFINITE PRODUCTS

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1. Introduction

Let D be the unit disk : $|z|< 1$ and D_0 the punctured disk $D-\{0\}\cup\{2^{-n}\}_{n=1}^{\infty}$ in which we consider a meromorphic function $f(z)=\Pi\{1+[2^m z-1]^{-1}\}$. Since $|2^{-m}(z-2^{-m})^{-1}|\leqq|2^{-m}(z-2^{-m_0})^{-1}|$ $(m=1,2,\ldots)$ for any z in D_0 and a certain integer m_0 which depends only on the point z, we can easily show that f(z) converges in D_0. When we take an arbitrary sequence $\{r_n\}$ such that $0<r_1<1/4$ and $0<r_{n+1}<2^{-n-1}-r_n$ $(n=1,2,\ldots)$, and denote by Dn the disk : $|z-2^{-n}|<r_n$, and further by $D(r_n)$ the region $D-\{0\}\cup\{\cup D_n\}$, our aim of this paper is to characterize $\{r_n\}$ so that the above function $f(z)$ might be bounded in $D(r_n)$.

2. A fundamental inequality

First, setting $\max\{|f(z)| : z\in\partial D_n\}=M_n$, we shall derive the following inequality :

(1) $k<M_n r_n 2^{\frac{n(n+1)}{2}}<(1+2^n r_n)^n\Pi\{1+[(1+2^n r_n)^{-1}2^j-1]^{-1}\}\Pi\{1+[(1-2^n r_n)2^j-1]^{-1}\}$

where k is a positive universal constant. Since

(2) $\log|f(z)|=\sum_{m\neq n}\log|z(z-2^{-m})|+\log|z|-\log r_n$ for z in ∂D_n,

and

(2)' $\log |z(z-2^{-m})|$ is harmonic in D_n, we get

$$(2\pi)^{-1} \int_{\partial D_n} \log |f(z)| dt = \sum \log |1-2^{n-m}| - n \log 2 - \log r_n.$$

Therefore, the definition of M_n implies that the right hand side of the above equality is less than $\log M_n$. Consequently we have

(3) $k < M_n r_n 2^n b_n^{-1}$

where $k = \prod_{m>n} |1+2^{-m}(2^{-n}-2^{-m})^{-1}| = \prod_{j\geqq 1} [1+(2^j-1)^{-1}]$

and $b_n = \prod_{m=1}^{n-1} (2^{n-m}-1)^{-1} = 2^{-\frac{n(n-1)}{2}} \prod_{j=1}^{n-1} (1+(2^j-1)^{-1})$.

Since $b_n > 2^{-\frac{n(n-1)}{2}}$, we get the first inequality appearing in (1) by virtue of (3).

Next, in order to obtain the second one, we observe that if z belongs to $\overline{D}_n$, the function $|z(z-2^{-m})^{-1}|$ ($m \geqq n+1$ or $m \leqq n$) attains its maximum at $z = 2^{-n}-r_n$ or $z = 2^{-n}+r_n$ respectively. Therefore we have

(4) $M_n < A_n B_n$ $(n=1,2,\ldots)$

where $A_n = \prod_{j\geqq 1} \{1+[(1-2^n r_n)2^j-1]^{-1}\}$

and

$$B_n = [r_n 2^{\frac{n(n+1)}{2}}]^{-1} (1+2^n r_n)^n \prod_{j=1}^{n-1} \{1+[(1+2^n r_n)^{-1} 2^j - 1]^{-1}\}.$$

This is exactly what we want to show. Now, before entering the proof of our theorem, we had better note that

$$\max [|f(z)| : |z| = 2^{-n}-r_n] = f(2^{-n}-r_n) \leqq M_n.$$

3. The main result

What we want to prove is the following

Theorem. $f(z) = \prod_{m=1}^{\infty} [1+(2^m z-1)^{-1}]$ is bounded in $D(r_n)$ if and only if $\underline{\inf}_n \, r_n 2^{\frac{n(n+1)}{2}} > 0$.

Proof. By virtue of (1), it is obvious that the boundedness of $f(z)$

in $D(r_n)$ implies $\inf r_n 2^{\frac{n(n+1)}{2}} > 0$. Conversely assume $\inf r_n 2^{\frac{n(n+1)}{2}} > 0$. And consider

$$r_n' = \min [r_n, \ 2^{-\frac{n(n+1)}{2}}]$$

and

$$M_n' = \max [|f(z)| : z \in \partial D_n']$$

where D_n' is the disk : $|z - 2^{-n}| < r_n'$. Then, making use of r_n' and M_n' in place of r_n and D_n used before, we get the corresponding inequality

(1)' $k < M_n' r_n' 2^{\frac{n(n+1)}{2}} < (1 + 2^n r_n') \prod_{j=1}^{n-1} \{1 + [(1 + 2^n r_n')2^j - 1]^{-1}\} \prod_{j=1}^{\infty} \{1 + [(1 - 2^n r_n')2^j - 1]^{-1}\}$.

Now, we remark here that an elementary inequality

(5) $\sum_{j=1}^{\infty} (2^j - a)^{-1} \leqq (4 - a)(4 - 2a)^{-1}$

holds for $0 < a < 2$. Then, our remaining task is to estimate from above the products appearing in (1)'. For this end, we note that $1 + 2^n r_n' \leqq 1/4$ and have only to see

$$\sum_{j=1}^{\infty} [(1 + 2^n r_n')^{-1} 2^j - 1]^{-1} \leqq 55/(24)$$

and

$$\sum_{j=1}^{\infty} [(1 - 2^n r_n')^{-1} 2^j - 1]^{-1} \leqq 8/3$$

which we can show easily by virtue of (5). This completes the proof.